A BÄCKLUND TRANSFORMATION BETWEEN 4D MARTÍNEZ ALONSO – SHABAT AND FERAPONTOV – KHUSNUTDINOVA EQUATIONS

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The aim of this note is to construct a Bäcklund transformation between the Lax-integrable 4-dimensional equations

$$u_{ty} = u_z\,u_{xy} - u_y\,u_{xz} \tag{1}$$

and

$$u_{yz} = u_{tx} + u_x\,u_{xy} - u_y\,u_{xx} \tag{2}$$

introduced in [4] and [1], respectively. Equation (2) has the following Lax pair [1]:

$$\left\{\begin{array}{lcl} v_t & = & \lambda\,v_y + u_y\,v_x, \\ v_z & = & (u_x+\lambda)\,v_x. \end{array}\right. \tag{3}$$

with non-removable parameter λ. Excluding u from (3) and normalizing $\lambda = 1$ in the resulting equation by re-scaling we get the equation

$$v_x\,v_{yz} = v_x\,v_{tx} + (v_y - v_t)\,v_{xx} - (v_x - v_z)\,v_{xy}. \tag{4}$$

Thus the differential covering (3) (in terms of [2]) defines a Bäcklund transformation between equations (2) and (4).

Proposition. *Equations (1) and (4) are point equivalent.*

Proof. Prolongation of the transformation $\psi\colon \mathbb{R}^4\times\mathbb{R} \to \mathbb{R}^4\times\mathbb{R}$

$$\psi(t,x,y,z,v) = (t,z,u,y+t,-x) \tag{5}$$

maps equation (4) to equation (1). Q.E.D.

Corollary. *The superposition of (5) and (3) defines a Bäcklund transformation between equations (1) and (2).*

Let us explain how we came to the existence of this transformation. We computed the contact symmetries of some integrable equations.

The infinite part of the symmetry group is parametrized by 3 copies of group $\mathrm{Diffeo}(\mathbb{R}^1)\,\hat{\otimes}\,C^\infty(\mathbb{R}^1)$ consisting of 1-parametric diffeomorphisms of $\mathbb{R}^1$ (can be changed to S^1); its Lie algebra is identified with the space of 1-parametric vector fields in 1D: $\mathscr{Q} = \{a(x,y)\partial_x\} \simeq$

$C^\infty(\mathbb{R}^2)$ (here $\mathbb{R}^2 = T\mathbb{R}^1$ can be changed to $TS^1 = S^1 \times \mathbb{R}^1$) with the Lie bracket

$$[a(x,y), b(x,y)] = a\,b_x - a_x b.$$

Everywhere below $\mathfrak{a}_i \simeq \mathscr{Q}$ will be the graded pieces of the infinite part $\mathfrak{s}_\infty$ of the symmetry algebra $\mathfrak{g}$. The upper index indicates the different commuting copies of the graded subalgebras, i.e. $[\mathfrak{g}_i^\alpha, \mathfrak{g}_j^\beta] \subset \delta^{\alpha\beta}\mathfrak{g}_{i+j}^\alpha$.

The symmetry algebra of equation (1) is the semi-direct product $\mathfrak{g} = \mathfrak{s}_\diamond \ltimes \mathfrak{s}_\infty$, where ($\mathfrak{a}_i \simeq \mathscr{Q}$)

$$\mathfrak{s}_\infty = (\mathfrak{a}_0' \oplus \mathfrak{a}_1') \oplus \mathfrak{a}_0''; \qquad \mathfrak{s}_\diamond = (\mathbb{R}^1 \ltimes \mathbb{R}^1) \oplus (\mathbb{R}^1 \ltimes \mathbb{R}^1).$$

Similarly, the symmetry algebra of equation (2) is the semi-direct product $\mathfrak{g} = \mathfrak{s}_\diamond \ltimes \mathfrak{s}_\infty$, where

$$\mathfrak{s}_\infty = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2; \qquad \mathfrak{s}_\diamond = \mathfrak{sl}_2 \ltimes (\mathbb{R}^2 \rtimes \mathbb{R}^1).$$

The modified (Bäcklund equivalent) version of equation (1) is the following equation, [5], (we again normalize $\lambda = 1$ by re-scaling)

$$u_x u_{ty} = (u_t + u_z)\,u_{xy} - u_y\,u_{xz}. \tag{6}$$

Its symmetry algebra is the semi-direct product $\mathfrak{g} = \mathfrak{s}_\diamond \ltimes \mathfrak{s}_\infty$, where

$$\mathfrak{s}_\infty = \mathfrak{a}_0' \oplus \mathfrak{a}_0'' \oplus \mathfrak{a}_0'''; \qquad \mathfrak{s}_\diamond = (\mathbb{R}^2 \rtimes \mathbb{R}^1);$$

The equations (1), (2), (6) have different symmetry algebras. But equation (4), which is a modified version of (2), has the same symmetry as (1). Although coincidence of symmetry algebras is only a necessary condition for equivalence of two equations, in this case we obtain the equivalence defined by (5).

Remark. In [3] we constructed symmetric integrable deformations of some heavenly type equations. These also exist for the equations (1), (2), (4) and (6) considered in this paper. For the first of them we have such an integrable deformation:

$$u_{ty} - u_z u_{xy} + u_y u_{xz} + (Q\,u_y)_y = 0, \quad Q = Q(t,y,z).$$

This equation has a covering defined by system

$$\left\{ \begin{array}{lcl} w_y &=& \lambda\, u_y\, w_x, \\ w_z &=& \lambda\,((u_z - \lambda\, Q\, u_y)\, w_x - w_t)\,. \end{array} \right.$$

The functional parameter Q is (essentially) non-removable. Details of this construction and information on other integrable deformations will appear in a forthcoming paper.

References

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